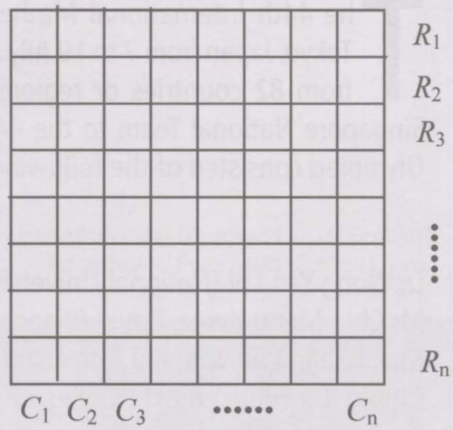
Mathematical Medley Problems Corner

A Prized Problems

Problem 1

(Book voucher up to $150)

Consider an n x n array with entries either 1 or -1. Let R_i be the product of the entries in row i and C_i be the product of the entries in column i ($1 \le i \le n$).



How many ways are there to arrange 1 and -1 in the boxes so that

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i \quad ?$$

Problem 2

(Book voucher up to $150)

How many solutions are there for the equation

$$1 = \frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_8}$$

where s_i are distinct numbers from the arithmetic progression $\{2,5,8,11,14,\ldots\}$.

B Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

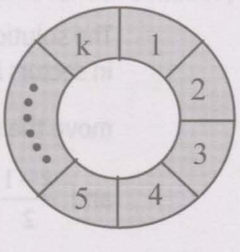
3. Solutions should be sent to :
The Editor, Mathematics Medley,
c/o Department of Mathematics,
National University of Singapore,
2 Science Drive 2, Singapore117543 ;
and should arrive before 1 May 2004.
Alternatively, softcopies of the solutions can also be sent to the email address: mattanv@nus.edu.sg.

4. The Editor's decision will be final and no correspondence will be entertained.

Solutions to the Problems of Volume 30, No 1, 2003

Problem 1

A ring is divided into k sectors (as shown in the diagram). A marble is placed in each sector. In any move, two marbles are shifted, one clockwise and the other anti-clockwise, into the adjacent sectors. (The two marbles being shifted need not come from the same sector.) Is it possible that, after a sequence of moves, all the k marbles end up in the same sector?



Solution 1 (Summing the sectors)

By Kenneth Tay Jingyi - ACS (Independent)

If a marble is in the i^{th} sector, let the marble's value be i.
Let S be the sum of all the values of all the marbles mod k.
If all the marbles end up in the same sector, for example the n^{th} sector, then

$$S \equiv nk \equiv 0 \pmod{k}.$$

Notice that after each move, S remains unchanged as the value of 1marble increases by 1 while the value of another marble decreases by 1.

Thus S is invariant, and so equal to its initial value $\frac{k(k+1)}{2}$. This is only congruent to 0 mod k if k is odd. (because $\frac{k+1}{2}$ is only an integer when k is odd.)
As such, it is possible that all the marbles end up in the same sector only if k is odd.

Now we shall show that if k is odd, then all the marbles can indeed end up in the same sector.
Let $k=2n+1$. Move the marble in the 1st sector clockwise and the marble in the $(2n+1)^{th}$ sector anti-clockwise simultaneously until they are both in the $(n+1)^{th}$ sector. Repeat this procedure for the marble in the ith sector and the marble in the $(2n+2-i)^{th}$ sector for $i = 2,3,\ldots,n$. It is easy to see that this procedure enables all the marbles to be moved into the same sector.

Solution 2 (Summing the moves)

By Zhao Yan - Raffles Institution

Without loss of generality, we can assume that the final sector where all the marbles will be in, if the problem is solvable, to be k, since the ring is symmetric.
We let each clockwise move be 1 and each anti-clockwise move be -1.
For k to be solvable, sum of all moves must be 0 since a clockwise and anticlockwise move are carried out at the same time.
If a marble is moved out and returned to the same sector, the sum of moves taken is 0 (mod k) since there will either be equal number of clockwise and anti-clockwise moves or k moves will be taken to move around the ring.
For a marble in sector i, to move it to sector k,
number of moves in clockwise direction $\equiv k - i \equiv -i \pmod{k}$
and number of moves in anti-clockwise direction $\equiv -i \pmod{k}$.

Therefore, sum of moves $\equiv \sum_{i=1}^{k}(-i) \equiv -\frac{k(k+1)}{2} \pmod{k}$.

If k is even, $-\frac{k(k+1)}{2} \equiv \frac{k}{2} \pmod{k}$.

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This is a contradiction and hence it is not possible to move all the marbles into one sector.

If k is odd, $-\frac{k(k+1)}{2} \equiv 0 \pmod{k}$ since $k+1$ is even.

The solution for k is odd is trivial. Leaving the marble in sector k untouched, we then move the marbles in sectors k-1 and 1 by one step in the clockwise and anti-clockwise direction respectively. Similarly, we move the marbles in sectors k-2 and 2 by two steps each and so on till we move the marbles in $\frac{k+1}{2}$ and $\frac{k-1}{2}$ by $\frac{k-1}{2}$ steps.

Hence, all the marbles can be moved into one sector if and only if k is odd.

Editor's note: Also solved correctly by Joel Tay Wei En (Raffles Junior College), Soh Yong Sheng (Raffles Institution), Cong Lin (Hwa Chong Junior College), and Andre Kueh Ju Lui (Chinese High School). The prize is shared by Kenneth Tay and Zhao Yan.

Problem 2

Find the smallest integer n such that

$$1 = \frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n}$$

where s_i are distinct numbers from the arithmetic progression $\{2,5,8,11,14,\ldots\}$.

Solution

By Joel Tay Wei En -Raffles Junior College

Obviously $n > 5$ as the 5 largest reciprocals of the series sum to

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} = \frac{3041}{3080} < 1$$

Note that $s_k \equiv 2 \pmod 3$ for all k.

If $n = 6$,

$$s_1s_2s_3s_4s_5s_6 = s_2s_3s_4s_5s_6 + s_1s_3s_4s_5s_6 + \cdots + s_1s_2s_3s_4s_5$$

which implies $1 \equiv 0 \pmod 3$, a contradiction.

Similarly, if $n = 7$,

$$s_1s_2s_3s_4s_5s_6s_7 = s_2s_3s_4s_5s_6s_7 + s_1s_3s_4s_5s_6s_7 + \cdots + s_1s_2s_3s_4s_5s_6$$

which implies $2 \equiv 1 \pmod 3$, a contradiction.

So n is at least 8. We observe that

$$s_1 = 2, s_2 = 5, s_3 = 8, s_4 = 14, s_5 = 20, s_6 = 35, s_7 = 56, s_8 = 140$$

yields a solution, so $n = 8$.

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More Solutions for the Equation with n=8

By Cong Lin - Hwa Chong Junior College

(Cong Lin's solution is similar to Joel's but he worked out a couple more solutions for the equation)

$$
\begin{aligned}
1 &= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{14}+\frac{1}{20}+\frac{1}{29}+\frac{1}{56}+\frac{1}{812} \\
&= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{14}+\frac{1}{20}+\frac{1}{44}+\frac{1}{56}+\frac{1}{77} \\
&= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{14}+\frac{1}{20}+\frac{1}{32}+\frac{1}{56}+\frac{1}{224} \\
&= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{20}+\frac{1}{44}+\frac{1}{152}+\frac{1}{209} \\
&= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{20}+\frac{1}{44}+\frac{1}{92}+\frac{1}{2024} \\
&= \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{20}+\frac{1}{44}+\frac{1}{104}+\frac{1}{572}
\end{aligned}
$$

Can you find more solutions? How many solutions are there all together? (See Prized Problem for this issue)

Editor's note: Also solved correctly by Li Zhipeng (Hwa Chong Junior College), Kenneth Tay Jingyi (ACS Independent), Ernest Chong Kai Fong (Raffles Junior College) and Zhao Yan (Raffles Institution). The prize is shared by Joel Tay and Cong Lin.